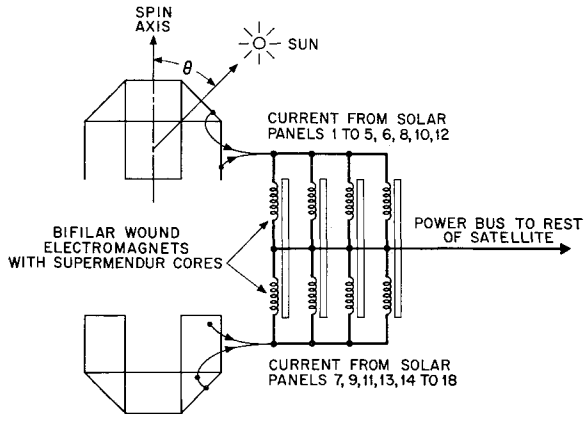


**Fig. 2 LES magnetic orientation system.**

However, if a core of very soft magnetic material with high intrinsic magnetization (such as Supermendur) is used instead of air cores, $M_{max}$ can be increased with no increase in power consumption or weight. Four electromagnetic coils with Supermendur cores 0.114 in. in diameter and 23 in. long, properly wound and spaced, theoretically can produce a maximum magnetic moment of 29-amp-m$^2$, and values of 18-amp-m$^2$ have been attained (the discrepancy is due to the difficulties in manufacturing Supermendur in this shape).

With the Supermendur cores, the variation of magnetic moment is no longer specified by Eq. (2), but $\Delta I$ is still given by Eq. (1). Instead of Eq. (2), we find that $\mathbf{M}$ is multivalued and depends nonlinearly on $\Delta I$. The value of $\mathbf{M}$ at any instant is found from the $B - H$ curve of Supermendur and the past values of $\Delta I$.

A first approximation for the dependence of magnetic moment on coil current is to assume that Supermendur has a perfectly square $B - H$ curve that has no hysteresis loss, and switches from $+B_{sat}$ to $-B_{sat}$ as $H$ changes sign. Then the variation of magnetic moment with orientation becomes

$$\mathbf{M} = +M_{sat}\mathbf{S} \qquad \mathbf{S}\cdot\mathbf{SOL} \geq 0 - M_{sat}\mathbf{S} \qquad \mathbf{S}\cdot\mathbf{SOL} < 0 \quad (3)$$

A schematic picture of the electromagnet arrangement is shown in Fig. 2.

The torque $\mathbf{T}$, produced by the interaction of this magnetic moment with the earth's geomagnetic field $\mathbf{B}$, is

$$\mathbf{T} = \dot{\mathbf{J}} = \mathbf{M} \times \mathbf{B}$$

where $\mathbf{J}$ is the satellite angular momentum. In the "fast-loop" approximation, we may solve this equation by directing the angular momentum along the spin axis at all times, thus, $= |\mathbf{J}|\mathbf{S}$ and

$$\mathbf{T} = \dot{\mathbf{J}} = |\mathbf{J}|\dot{\mathbf{S}} = \mathbf{M} \times \mathbf{B} = \mp M_{max}(\mathbf{S} \times \mathbf{B}) \qquad \mathbf{S}\cdot\mathbf{SOL} \gtrless 0$$

The solution of this equation is characterized by a precession of the spin axis $\mathbf{S}$ around the instantaneous value of $\mathbf{B}$, with the details of the motion dependent on $\mathbf{B}$ and the solar vector. A 7094 computer program was written to calculate the value of $\mathbf{B}$ (using a 11.4° tilted dipole approximation for the earth's field) as the satellite moves in its orbit; the solar vector; and the vector relationships necessary to perform a numerical integration of the spin-axis motion, using Eq. (3) to specify the magnetic moment.

The results of this program indicate that the satellite to be launched early in 1965 with 1.5K- to 10.0K-naut-mile orbit inclined 32° to the equator (eccentricity $\cong$ 0.53, period $\cong$ 7.5 hr) will experience a maximum precessional rate of about 5°/day around the average magnetic-field vector (which will be inclined about 23° to the orbit normal). Thus, if the satellite were to be launched so that the spin vector pointed into (or away from) the sun initially, it would precess to perpendicularity in about 20 days and maintain this orientation thereafter, as the earth and satellite orbit the sun.

# Newton-Raphson Operator; Problems with Undetermined End Points

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IN a recent paper, McGill and Kenneth[1] have successfully applied the technique of quasi-linearization (a generalized Newton-Raphson operator) to the numerical solution of certain optimization problems. These involve determining the solution of a set of nonlinear differential equations with boundary conditions specified at two points, i.e., at two values of the independent variable $t$, namely $t = 0$ and $t = t_f$. However, in one of their examples, the value of $t_f$ is unknown, and so must be determined as part of the whole iterative process.

In the paper quoted, the method given for dealing with $t_f$ appears to suffer from two disadvantages. First, it is not an integral part of the quasi-linearization process. In what may be called a normal quasi-linearization process, all the boundary conditions are exactly satisfied at each iteration (apart from roundoff and truncation error). Presumably it is this feature of a normal process which helps to explain its excellent convergence. In the method given, however, one of the boundary conditions is used to modify $t_f$, and this boundary condition is not satisfied until convergence has actually occurred. This feature may be expected to slow down the rate of convergence. Second, the method given depends on the numerical calculation of a derivative, a procedure to be avoided if possible.

The purpose of this note is to present an alternative method for dealing with $t_f$ which avoids both of the forementioned disadvantages. We make a change of variable by writing $t = as$ where $s$ is the new independent variable, and $a$ is a constant to be determined. We take the end points to be $s = 0$, $s = 1$, so that, once $a$ is found, the value of $t_f$ is given by $t_f = a$. Suppose that a typical differential equation of the system is $\dot{x} = f$, where $f$ is, in general, a function of all the dependent variables, and also of $t$, and $\dot{x} = dx/dt$. Writing $dx/ds = x'$, we have $x' = (dx/dt)(dt/ds) = a\dot{x}$ so that $x' = af$ where, in $f$, $t$ is replaced by $as$. We now treat $a$ as an additional dependent, or state, variable, and include it in the quasi-linearization process. Then the set of equations is of exactly the same form as before, and there will be the correct number of boundary conditions. The equations are slightly more complicated than before, but not unduly so. Of course, $a$ is a constant during each iteration, but, like the other state variables, it varies from one iteration to the next. This treatment of $a$ as a state variable may be formalized by writing $a' = 0$ and by adding this equation to the system, but, from a computational point of view, this is not necessary.

Because of the excellent convergency properties of the quasi-linearization process, and because the determination of $t_f$ is now an integral part of that process, it is to be expected that the total computation time will be reduced as compared with the time for the previous method. This, however, is a matter for numerical experiment.

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There is the further advantage in that the integration interval from $s = 0$ to $s = 1$ is now fixed, thus simplifying the calculation of the coefficients in the linearized equations since these coefficients must be calculated at the end of each integration step.

The method given here can be generalized. Thus consider the case of a three-point boundary problem, involving say $t = 0$, $t = t_1$, $t = t_2$, where $t_1$ and $t_2$ are unknown. Such a situation can arise in rocket problems where it is necessary to switch from one mode of operation to another at time $t_1$. Then we write $t = as$, $0 \leqslant s \leqslant 1$, $t = a + b(s - 1)$, and $1 \leqslant s \leqslant 2$, with the boundary points given by $s = 0, 1, 2$, so that $t_1 = a$ and $t_2 = a + b$. It is necessary that $t$ be a continuous, increasing function of $s$. Clearly the first of these conditions is satisfied. Also, since $0 \leqslant t_1 \leqslant t_2$, it follows that $a \geqslant 0$, $b \geqslant 0$, so that $t$ is also an increasing function of $s$. Using the same method as before, and using an obvious convention concerning left-hand and right-hand derivatives at the end points of the intervals, the equation $\dot{x} = f$ becomes $x' = af$, $0 \leqslant s \leqslant 1$ and $x' = bf$, $1 \leqslant s \leqslant 2$; we treat $a$ and $b$ as state variables. The derivative $x'$ now has a simple discontinuity at $s = 1$, but this does not affect the existence or uniqueness of the solution of the system and presents no difficulty in programing for a computer. Clearly this method can be extended to any number $n + 1$ of unknown boundary points $t_0, t_1, \ldots, t_n$ by taking the corresponding points as $s = 0, 1, \ldots, n$ and writing $t = a_1 + b_1 s$, $0 \leqslant s \leqslant 1$, $t = a_{r+1} + b_{r+1}(s - r)$, $r \leqslant s \leqslant r + 1$, and $r = 1, 2, \ldots, n - 1$. The continuity conditions are $a_{r+1} = a_r + b_r$ and $r = 1, 2, \ldots, n - 1$, and the values of $t$ at the boundary points are given by $t_0 = a_1$, $t_r = a_r + b_r$, and $r = 1, 2, \ldots, n$.

### Numerical Example

As an example we consider the system of equations $\ddot{x} = -x/r^3$ and $\ddot{y} = -y/r^3$ where $r^2 = x^2 + y^2$. The given boundary values are taken as

$$x(0) = 1.0 \qquad y(0) = 0.0$$
$$x(t_1) = 0.921061 \qquad y(t_1) = 0.389418$$
$$x(t_2) = 0.540303 \qquad y(t_2) = 0.841471$$

where both $t_1$ and $t_2$ are taken as unknown. [These values were in fact predetermined by taking the initial values $x(0) = 1.0$, $\dot{x}(0) = 0.0$, $y(0) = 0.0$, $\dot{y}(0) = 1.0$ together with $t_1 = 0.4$, $t_2 = 1.0$.] Applying the preceding method, we write $\dot{x} = u$, $\dot{y} = v$, and $t = as$ in $0 \leqslant s \leqslant 1$, $t = a + b(s - 1)$ in $1 \leqslant s \leqslant 2$, and so obtain the equations $x' = au$, $y' = av$, $u' = -ax/r^3$, and $v' = -ay/r^3$ with $b$ replacing $a$ in $1 \leqslant s \leqslant 2$. We now have six independent variables $x$, $y$, $u$, $v$, $a$, and $b$, and the linearization process is carried out with respect to these, in the manner explained in Ref. 1.

Starting functions were obtained for $x$ and $y$ by linear interpolation of the known values at $t = 0$, $t = t_2$ over the interval $0 \leqslant s \leqslant 2$. Values of $t_1$, $t_2$, and, hence, of $a$ and $b$, were assumed, and corresponding values for $u$ and $v$ were assumed constant over the whole interval. For the seven forward integrations required for each iteration, initial vectors of the type (1, 0, 0, 0, 0, 0) were found satisfactory. A fourth-order Runge-Kutta scheme was used which, at the end of each iteration, required that the values of the variables at the midpoint of each integration step be calculated and be ready to provide the coefficients for the next iteration. These values were obtained by interpolation.

**Table 1 Values of variables from $k$ th iteration.**

| $k$ | $u(0)$ | $v(0)$ | $t_1$ | $t_2$ |
|---|---|---|---|---|
| 0 | −0.8 | 1.2 | 0.3 | 0.7 |
| 1 | 0.50802 | 0.42421 | 0.52782 | 1.28177 |
| 2 | 0.10588 | 0.86884 | 0.38348 | 1.05994 |
| 3 | −0.00123 | 0.99448 | 0.40521 | 0.99043 |
| 4 | −0.00016 | 1.00016 | 0.39990 | 0.99982 |
| 5 | −0.00002 | 1.00000 | 0.39999 | 0.99999 |
| ... | 0.0 | 1.0 | 0.4 | 1.0 |

The following results were obtained with a step size in $s$ of 0.1. Six digits were used in the calculations, and the results are shown to five decimal places. Table 1 shows the values of the indicated variables, obtained from the $k$th iteration. $x(0)$, $y(0)$ retain their given values and so are not shown. The first row ($k = 0$) contains the initial, assumed values, and the last row shows the true values.

In conclusion, it would appear that the method suggested affords a practical, efficient means of applying quasi-linearization to problems with undetermined end points.

### Reference

[1] McGill, R. and Kenneth, P., "Solution of variational problems by means of a generalized Newton-Raphson operator," AIAA J. 2, 1761–1766 (1964).

# Torsion of an Aeolotropic Cylinder Having a Spheroidal Inclusion on its Axis

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### Introduction

RECENTLY, Bhowmick[1] has solved the problem of torsion of a circular cylinder of transversely isotropic material having a rigid spheroidal inclusion on its axis. Here we shall give the stresses due to an elastic inclusion in the form of a spheroid situated symmetrically on the axis of a large transversely isotropic circular cylinder under torsion.

### Equations and Boundary Conditions

We take the origin at the center of the inclusion and the $z$ axis along the axis of symmetry of the inclusion with the axis of the cylinder coinciding with the axis of figure of the inclusion. Let $(r, \theta, z)$ be the cylindrical coordinates of any point, and let $(u, v, w)$, $(u', v', w')$ be, respectively, the components of displacements outside the inclusion and internal to it in the increasing directions of $r$, $\theta$, and $z$, respectively. The cylinder is assumed to be twisted about $z$ axis. In the sequel, primed quantities will refer to the material of the inclusion.

When the material is under torsion, we assume

$$u = 0 = w \qquad v = v(r, z) \tag{1}$$
$$u' = 0 = w' \qquad v' = v'(r, z)$$

Then nonzero strain components are given by

$$e_{r\theta} = r\Phi_{,r} \qquad e_{\theta z} = r\Phi_{,z}$$
$$e_{r\theta}' = r\Phi_{,r}' \qquad e_{\theta z}' = r\Phi_{,z}' \tag{2}$$

where

$$\Phi = v/r \qquad \Phi' = v'/r \qquad \Phi_{,r} \equiv \partial\Phi/\partial r$$
$$\Phi_{,z} \equiv \partial\Phi/\partial z \qquad \text{etc.}$$

Nonzero stress components inside the inclusion are

$$p_{r\theta}' = Gr\Phi_{,r}' \qquad p_{\theta z}' = Gr\Phi_{,z}' \tag{3}$$

where $G$ is the modulus of rigidity of the inclusion.

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